

1) a) $\vec{v} = \langle axy, \sin(xy), z \rangle$. $\nabla \cdot \vec{v} = ay + x \cos(xy) + 1$.

$\nabla \cdot \vec{v} \Big|_{(y=\pi/4, -1)} = a \frac{\pi}{4} + \frac{\sqrt{2}}{2} - 1 = 0 \Rightarrow a = \frac{4}{\pi} (1 - \frac{\sqrt{2}}{2})$.

b) $\nabla \times \langle y-x, -x, 0 \rangle = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-x & -x & 0 \end{bmatrix} = \langle 0, 0, -1-1 \rangle = -2\hat{k}$.

Flow is rotational.

c) $\nabla \times \vec{v} = \langle \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \rangle$

$$\begin{aligned} \text{div}(\nabla \times \vec{v}) &= \frac{\partial}{\partial x} \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \\ &= \underbrace{\frac{\partial^2 v_3}{\partial x \partial y}} - \underbrace{\frac{\partial^2 v_2}{\partial x \partial z}} + \underbrace{\frac{\partial^2 v_1}{\partial y \partial z}} - \underbrace{\frac{\partial^2 v_3}{\partial y \partial x}} + \underbrace{\frac{\partial^2 v_2}{\partial z \partial x}} - \underbrace{\frac{\partial^2 v_1}{\partial z \partial y}} = 0 \end{aligned}$$

if \vec{v} is a smooth vector field.

2) a) $u(x,y) = \sin x \cos 2y$. By definition, u is an eigenfunction of L if $Lu = \lambda u$ for some number λ .

$$\begin{aligned} \underbrace{\left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right)}_L (\underbrace{\sin x \cos 2y}_u) &= -\frac{\partial^2}{\partial x^2} (\sin x \cos 2y) - \frac{\partial^2}{\partial y^2} (\sin x \cos 2y) \\ &= \sin x \cos 2y + 4 \sin x \cos 2y = 5 \sin x \cos 2y \\ &= 5u \end{aligned}$$

So, u is an eigenfunction with eigenvalue 5.

b) $F'' + F' + 3F = 0$. Try $F(x) = e^{mx}$. m satisfies $m^2 + m + 3 = 0$,

so $m = \frac{-1 \pm \sqrt{1-12}}{2} = \frac{-1 \pm \sqrt{11}i}{2}$. But

$$e^{(-\frac{1}{2} \pm \frac{\sqrt{11}i}{2})x} = e^{-\frac{1}{2}x} \left(\cos \frac{\sqrt{11}}{2}x + i \sin \frac{\sqrt{11}}{2}x \right)$$

So, the general solution of $F'' + F' + 3F = 0$ is

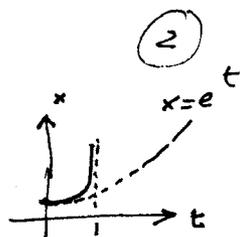
$$F(x) = c_1 e^{-\frac{1}{2}x} \cos \frac{\sqrt{11}}{2}x + c_2 e^{-\frac{1}{2}x} \sin \frac{\sqrt{11}}{2}x.$$

2c)

$$\frac{dx}{dt} = x^2 \Rightarrow \frac{dx}{x^2} = dt \Rightarrow -\frac{1}{x} = t + c \Rightarrow$$

$$x = -\frac{1}{t+c}. \quad x(0) = 1 \Rightarrow 1 = -\frac{1}{c} \Rightarrow c = -1. \quad \text{So}$$

$$x(t) = -\frac{1}{t-1}. \quad \lim_{t \rightarrow 1} x(t) = \infty.$$



$\frac{dx}{dt} = x$, $x(0) = 1 \Rightarrow x(t) = e^t$. This solution becomes unbounded only as $t \rightarrow \infty$. The difference in behavior of these two differential equations has to do with the first equation, $x' = x^2$, being nonlinear.

3) a)

$Ax + By + Cz = D$ is the equation of the plane. Substitute $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 0)$ into this equation to get:

$$\begin{cases} A + C = D \\ B + C = D \\ A + B = D \end{cases} \Rightarrow \begin{cases} B + C = D \\ B - C = 0 \end{cases} \Rightarrow \begin{cases} B = C = D/2 \\ A = D/2 \end{cases}$$

So $\frac{D}{2}x + \frac{D}{2}y + \frac{D}{2}z = D \Rightarrow \boxed{\frac{x}{2} + \frac{y}{2} + \frac{z}{2} = 1}$. So

$z = 2 - x - y$. Parametrization

$$\boxed{\vec{r}(x, y) = \langle x, y, 2 - x - y \rangle.}$$

b) $\vec{r}(\theta, \phi) = \langle R \cos \theta \cos \phi, R \sin \theta \cos \phi, R \sin \phi \rangle.$

$$\vec{r}_\theta \times \vec{r}_\phi = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -R \sin \theta \cos \phi & R \cos \theta \cos \phi & 0 \\ R \cos \theta \sin \phi & -R \sin \theta \sin \phi & R \cos \phi \end{bmatrix}$$

$$= \langle R^2 \cos \theta \cos^2 \phi, R^2 \sin \theta \cos^2 \phi, R^2 \cos \phi \sin \phi \rangle$$

$$\begin{aligned} \vec{N} &= \vec{r}_\theta \times \vec{r}_\phi \Big|_{\left(\frac{\pi}{3}, \frac{\pi}{6}\right)} = R^2 \left\langle \cos \frac{\pi}{3} \cos^2 \frac{\pi}{6}, \sin \frac{\pi}{3} \cos^2 \frac{\pi}{6}, \cos \frac{\pi}{3} \sin \frac{\pi}{6} \right\rangle \\ &= R^2 \left\langle \frac{1}{2} \left(\frac{\sqrt{3}}{2}\right)^2, \frac{\sqrt{3}}{2} \left(\frac{\sqrt{3}}{2}\right)^2, \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \right\rangle \\ &= R^2 \left\langle \frac{3}{8}, \frac{3\sqrt{3}}{8}, \frac{\sqrt{3}}{4} \right\rangle \end{aligned}$$

$$\|\vec{N}\| = R^2 \sqrt{\frac{9}{64} + \frac{27}{64} + \frac{3}{16}}$$

$$= R^2 \sqrt{\frac{48}{64}} = R^2 \frac{\sqrt{3}}{2}$$

unit normal vector = $\frac{2}{2\sqrt{3}} \vec{N} = \frac{2}{2\sqrt{3}} \left\langle \frac{3}{8}, \frac{3\sqrt{3}}{8}, \frac{\sqrt{3}}{4} \right\rangle.$

4) (a) $\phi = axy^2 - bx^2y + x.$

$\vec{v} = \nabla \phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle = \langle ay^2 - 2bxy + 1, 2ax - bx^2 \rangle$

$\vec{v}|_{(2,-1)} = \langle a + 4b + 1, 4a - 4b \rangle = \langle 0, 0 \rangle \Rightarrow$

$$\begin{cases} a + 4b + 1 = 0 & \Rightarrow b = -1/5 \\ 4a - 4b = 0 & \Rightarrow a = b \end{cases} \Rightarrow a = -1/5.$$

(b) $\psi(x,y) = ax^2 + xy + by^2.$

$\vec{v} = \left\langle \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right\rangle = \langle x + 2by, -2ax - y \rangle.$

$\vec{v}|_{(1,-1)} = \langle 1 - 2b, -2a + 1 \rangle.$

$\|\vec{v}|_{(1,-1)}\| = \sqrt{(1-2b)^2 + (-2a+1)^2} = \frac{1}{3} \Rightarrow$

$(1-2b)^2 + (-2a+1)^2 = \frac{1}{9} \Rightarrow$

$4b^2 - 4b + 4a^2 - 4a = \frac{1}{9} - 1 - 1 = -\frac{17}{9}.$

$$b^2 - b + a^2 - a = -\frac{17}{36}$$



5) (a) $\iint_S \vec{v} \cdot d\vec{A} = \iint_D \langle y^2, 0, x^2z \rangle \Big|_S \cdot \vec{r}_u \times \vec{r}_v \, du \, dv.$

$\vec{r}(u,v) = \langle u \cos v, u \sin v, 0 \rangle; \vec{r}_u \times \vec{r}_v = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{bmatrix}$

$= \langle 0, 0, u \rangle$

so $\iint_D \langle y^2, 0, x^2z \rangle \Big|_S \cdot \vec{r}_u \times \vec{r}_v \, du \, dv$

$= \int_0^{2\pi} \int_0^1 \langle u^2 \sin^2 v, 0, 0 \rangle \cdot \langle 0, 0, u \rangle \, du \, dv = 0.$

(b) $\vec{r}(u,v) = \langle u \cos v, u \sin v, 1 \rangle. \vec{r}_u \times \vec{r}_v = \langle 0, 0, u \rangle.$

$\iint_D \langle y^2, 0, x^2z \rangle \Big|_S \cdot \vec{r}_u \times \vec{r}_v \, du \, dv = \int_0^{2\pi} \int_0^1 \langle u^2 \sin^2 v, 0, u^2 \cos^2 v \rangle \cdot \langle 0, 0, u \rangle \, du \, dv$

$$\int_0^{2\pi} \int_0^1 u^3 \cos^2 v \, du \, dv = \int_0^{2\pi} \cos^2 v \left(\frac{u^4}{4} \right) \Big|_{u=0}^{u=1} \, dv = \frac{1}{4} \int_0^{2\pi} \cos^2 v \, dv$$

$$= \frac{1}{8}$$

6) $\iint_S \nabla \times \vec{v} \cdot d\vec{A} = \oint_C \vec{v} \cdot d\vec{r}$

C: ellipse $2x^2 + 3y^2 = 1$.

Parametrize: $2x^2 = \cos^2 t, \quad 3y^2 = \sin^2 t. \Rightarrow$

$$\vec{r}(t) = \left\langle \frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{3}} \sin t, 0 \right\rangle$$

~~$$\frac{d\vec{r}}{dt} = \left\langle -\frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{3}} \cos t, 0 \right\rangle$$~~

$$\oint_C \vec{v} \cdot d\vec{r} = \int_0^{2\pi} \vec{v} \Big|_C \cdot \frac{d\vec{r}}{dt} \, dt = \int_0^{2\pi} \left\langle \frac{1}{3} \cos^2 t, 0, 0 \right\rangle \cdot \left\langle -\frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{3}} \cos t, 0 \right\rangle dt$$

$$= \int_0^{2\pi} -\frac{1}{\sqrt{6}} \cos^2 t \sin t \, dt = \left. \frac{1}{3\sqrt{6}} \cos^3 t \right|_0^{2\pi} = 0$$

7) $u_t = 4u_{xx}, \quad u(x,t) = F(x)G(t) \Rightarrow u_t = FG', \quad u_{xx} = F''G$

So $FG' = 4F''G \Rightarrow$

$$\frac{F''}{F} = \frac{G'}{4G} = -\lambda^2 \Rightarrow \begin{cases} F'' + \lambda^2 F = 0 \Rightarrow F(x) = c_1 \sin \lambda x + c_2 \cos \lambda x \\ G' + 4\lambda^2 G = 0 \Rightarrow G(t) = e^{-4\lambda^2 t} \end{cases}$$

So $u(x,t) = e^{-4\lambda^2 t} (c_1 \sin \lambda x + c_2 \cos \lambda x)$

Now, $u(0,t) = 0 = e^{-4\lambda^2 t} (c_2) \Rightarrow c_2 = 0 \Rightarrow u(x,t) = c_1 e^{-4\lambda^2 t} \sin \lambda x$

$u(2,t) = 0 = c_1 e^{-4\lambda^2 t} \sin 2\lambda \Rightarrow \sin 2\lambda = 0 \Rightarrow 2\lambda = n\pi \Rightarrow \lambda = \frac{n\pi}{2}$

So $u(x,t) = c_n e^{-n^2 \pi^2 t} \sin \frac{n\pi x}{2}$

General solution: $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 t} \sin \frac{n\pi x}{2}$

I.C $\Rightarrow x(2-x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{2} \Rightarrow$

$$c_n = \frac{(x(2-x), \sin \frac{n\pi x}{2})}{(\sin \frac{n\pi x}{2}, \sin \frac{n\pi x}{2})} = \frac{\int_0^2 x(2-x) \sin \frac{n\pi x}{2} \, dx}{\int_0^2 \sin^2 \frac{n\pi x}{2} \, dx}$$

$c_1 = \frac{32}{3}$

At $t=0$ the temperature at $x=1.5$ is $1.5(2-1.5) = 0.75$.

$u(x,t) = \frac{32}{\pi^3} e^{-\pi^2 t} \sin \frac{\pi x}{2}$ if we use one term of the series solution.

We need to find t so that

$u(1.5,t) = \frac{32}{\pi^3} e^{-\pi^2 t} \sin \frac{1.5\pi}{2}$

equals 0.75. Using the calculator, we find

$t = -\frac{1}{\pi^2} \ln\left(\frac{0.75 \pi^3 \sin 1.5\pi/2}{32}\right) = 0.067$.

8) $\vec{v} = \langle x^2+y^2, 2xy \rangle \Rightarrow x' = x^2+y^2, y' = 2xy$, where ' mean $\frac{d}{dt}$.

$v_1 = x^2+y^2 \Rightarrow a_1 = 2xx' + 2yy' = 2x(x^2+y^2) + 2y(2xy) = 2x^3 + 2xy^2 + 4xy^2 = 2x^3 + 6xy^2$.

$v_2 = 2xy \Rightarrow a_2 = 2x'y + 2xy' = 2(x^2+y^2)y + 2x(2xy) = 2x^2y + 2y^3 + 4x^2y = 6x^2y + 2y^3$.

$\vec{a} = \langle 2x^3 + 6xy^2, 6x^2y + 2y^3 \rangle$.

9) a) $\Omega = 2\pi \text{ radians} / (24 \text{ (hours)} \times 60 \text{ (minutes)} \times 60 \text{ (seconds)}) = .00007 \text{ rad/sec}$.

b) $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$

$\vec{\Omega} \times \vec{r} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega \\ x & y & z \end{bmatrix} = \langle -y\Omega, -x\Omega, 0 \rangle$

$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega \\ -y\Omega & -x\Omega & 0 \end{bmatrix} = \langle -x\Omega^2, -y\Omega^2, 0 \rangle = -\Omega^2 \langle x, y, 0 \rangle$.

Take the "typical" point at the equator. Then $\|\langle x, y, 0 \rangle\| = 6000 \text{ kilometers}$.

So $\|\vec{\Omega} \times (\vec{\Omega} \times \vec{r})\| \leq 6000 \text{ (kilometers)} \times 1000 \text{ (meters)} \times (.00007 \text{ rad/s})^2 \approx .03 \text{ m/s}^2$.

$$c) \quad -fv = -\frac{1}{f} \frac{\partial P}{\partial x}, \quad fu = -\frac{1}{f} \frac{\partial P}{\partial y}$$

$$\Rightarrow u = -\frac{1}{f f} \frac{\partial P}{\partial y}, \quad v = \frac{1}{f f} \frac{\partial P}{\partial x}$$

$$\vec{v} = \langle u, v \rangle = \frac{1}{f f} \left\langle -\frac{\partial P}{\partial y}, \frac{\partial P}{\partial x} \right\rangle$$

$$a) \quad \nabla \cdot \vec{v} = \frac{1}{f f} \left[\frac{\partial}{\partial x} \left(-\frac{\partial P}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{\partial P}{\partial x} \right) \right] = \frac{1}{f f} \left[-\frac{\partial^2 P}{\partial x \partial y} + \frac{\partial^2 P}{\partial y \partial x} \right] = 0.$$

b) Paths of particles are determined by solving the system of differential equations

$$\frac{dx}{dt} = u = -\frac{1}{f f} \frac{\partial P}{\partial y}, \quad \frac{dy}{dt} = \frac{1}{f f} \frac{\partial P}{\partial x}.$$

On the other hand, $\nabla P \perp P = \text{const.} = \text{isobar}$ by the fundamental theorem we have established in class. However, in this special setting

$$\nabla P \cdot \vec{v} = \left\langle \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y} \right\rangle \cdot \left\langle -\frac{1}{f f} \frac{\partial P}{\partial y}, \frac{1}{f f} \frac{\partial P}{\partial x} \right\rangle$$

$$= -\frac{1}{f f} \frac{\partial P}{\partial x} \frac{\partial P}{\partial y} + \frac{1}{f f} \frac{\partial P}{\partial y} \frac{\partial P}{\partial x} = 0.$$

So $\nabla P \perp \vec{v}$. Since $\nabla P \perp P = \text{const.} \Rightarrow$ ~~isobar~~

\vec{v} is parallel with $P = \text{const.} \Rightarrow$ particle paths

and isobars coincide.

c) In northern hemisphere $f > 0$. At a high pressure point A , $\frac{\partial P}{\partial x} = 0$ and $\frac{\partial P}{\partial y} > 0$.

Let compute v , the vertical component of velocity, at a point to the left of H (marked by A). Since P must become max at H , then

$$\left. \frac{\partial P}{\partial x} \right|_A > 0. \text{ Then}$$

$$v = \frac{1}{f f} \left. \frac{\partial P}{\partial x} \right|_A > 0$$

Therefore, the motion around H will be clockwise.

